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Gödel–Rosser's Incompleteness Theorems
for Non–Recursively Enumerable Theories

Abstract

Gödel's First Incompleteness Theorem is generalized to definable theories, which are not necessarily recursively enumerable, by using a couple of syntactic-semantic notions; one is the consistency of a theory with the set of all true Π_n-sentences or equivalently the Σ_n-soundness of the theory, and the other is n-consistency the restriction of ω-consistency to the Σ_n-formulas. It is also shown that Rosser's Incompleteness Theorem does not generally hold for definable non-recursively enumerable theories; whence Gödel-Rosser's Incompleteness Theorem is optimal in a sense. Though the proof of the incompleteness theorem using the Σ_n-soundness assumption is constructive, it is shown that there is no constructive proof for the incompleteness theorem using the n-consistency assumption, for $n > 2$.

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1   Introduction and Preliminaries

Gödel's First Incompleteness Theorem is usually taken to be *the incompleteness of the first order theory of Peano Arithmetic PA*. While *PA* is not a complete theory, the theorem states much more than that. One of the most misleading ways for stating the theorem is: *any sound theory containing PA is incomplete*, where a theory is called sound when all its axioms are true in the standard model of natural numbers $\mathbb{N}$. A quick counterexample for this statement, often asked by new learners of the incompleteness, is that *but the theory of true arithmetic* $\mathrm{Th}(\mathbb{N})$ *is complete?!*, where $\mathrm{Th}(\mathbb{N})$ is the set of sentences that are true in the standard model of natural numbers. Of course, the obvious answer is that $\mathrm{Th}(\mathbb{N})$ *is not recursively enumerable* (RE for short). So, the right rewording of Gödel's First Incompleteness Theorem in its (weaker) semantic form is that *any sound and* RE *theory containing PA is incomplete*. Now, a natural second question is: *what about non-*RE *theories* (*that are sound and contain PA*)? Again the same obvious answer shows up: $\mathrm{Th}(\mathbb{N})$ *is not* RE (by the very theorem of Gödel's first incompleteness) *and is complete*. So, the question of the incompleteness of non-RE theories should come down to more specific ones, at least to finitely representable theories, or, as the logicians say, definable ones. Hence, *do we have the incompleteness of definable theories* (*which are sound and contain PA*)? This question has been answered affirmatively in the literature; see e.g. [15] or [12]. Gödel's original first incompleteness theorem did not assume the soundness of the theory in question, and he used the notion of ω-consistency for that purpose. Later it was found out that the weaker notion of 1-consistency suffices for the theorem (see e.g. [3] or [13]). By generalizing this equivalent notion to higher degrees (Π_n in general) we will prove some generalizations of Gödel's first incompleteness theorem for definable theories below. Finally, Rosser's Trick proves Gödel's result without assuming the 1-consistency of the theory. So, Gödel-Rosser's Incompleteness Theorem, assuming only the consistency of the theory, states that *any consistent and* RE *theory containing PA is incomplete*. It is tempting to weaken the condition of recursive enumerability of the theory in this theorem; but we will see below that this is not possible. We can thus argue that Gödel-Rosser's theorem is optimal in a sense.

1.1   Some Notation and Conventions

We fix the following notation and conventions (mostly from [2, 3, 6, 13, 15]). Fix a language of arithmetic, like $\{0, S, +, \times, \leqslant\}$ (as in [2]) or $\{0, 1, +, \times, <\}$ (as in [6]).

- For any natural number $n \in \mathbb{N}$ the term $\overline{n}$ represents this number in the fixed arithmetical language (which could be $S \cdots S(0)$ or $1 + \cdots + 1$ [n-times]). For a fixed Gödel numbering of syntax, $\ulcorner \alpha \urcorner$ denotes the Gödel number of the object α; when there is no ambiguity we will write simply $\ulcorner \alpha \urcorner$ for the term $\overline{\ulcorner \alpha \urcorner}$. Any Gödel numbering consists of coding sequences; if m is the code of a sequence, then the formula $\mathsf{Seq}(m)$ expresses this fact, and its length is denoted by $\ell en(m)$ and for any number $l < \ell en(m)$ the lth member of m is denoted by $[m]_l$. A sequence m is thus $\langle [m]_0, [m]_1, \cdots, [m]_{\ell en(m)-1} \rangle$; and for any $k \leqslant \ell en(m)$, the initial segment of m with length k is denoted by $\langle m \restriction k \rangle$, that is $\langle [m]_0, [m]_1, \cdots, [m]_{k-1} \rangle$. Note that $\langle m \restriction 0 \rangle = \emptyset$ and $\langle m \restriction \ell en(m) \rangle = m$. If m is the Gödel code of a sentence, then $\mathsf{Sent}(m)$ expresses this fact. For a sequence of sentences like m, the formula $\mathsf{ConjSeq}(k, m)$ means that "k is the (Gödel code of the) conjunction of all the members of m", i.e., $k = \ulcorner \bigwedge_{i < \ell en(m)} \varphi_i \urcorner$ where $[m]_i = \ulcorner \varphi_i \urcorner$. The propositional connectives may act (as numeral partial functions) on natural numbers; for example $\neg m$ for $m \in \mathbb{N}$ is $\ulcorner \neg \alpha \urcorner$ where $m = \ulcorner \alpha \urcorner$, and for any $\circ \in \{\wedge, \vee, \rightarrow\}$ and $m, k \in \mathbb{N}$, $m \circ k = \ulcorner \alpha \circ \beta \urcorner$ where $m = \ulcorner \alpha \urcorner$ and $k = \ulcorner \beta \urcorner$.

- The classes of formulas $\{\Sigma_n\}_{n \in \mathbb{N}}$ and $\{\Pi_n\}_{n \in \mathbb{N}}$ are defined in the standard way [2, 6]: $\Sigma_0 = \Pi_0$ is the class of bounded formulas (in which every universal quantifier has the form $\forall x([x \leqslant t \rightarrow \cdots]$ and every existential quantifier has the form $\exists x[x \leqslant t \wedge \cdots])$, and the class Σ_{n+1} contains the closure of Π_n

under the existential quantifiers, and is closed under disjunction, conjunction, existential quantifiers and bounded universal quantifiers; similarly, the class Π_{n+1} contains the closure of Σ_n under the universal quantifiers, and is closed under disjunction, conjunction, universal quantifiers and bounded existential quantifiers. By definition $\Delta_n = \Sigma_n \cap \Pi_n$. Let us note that the negation of a Σ_n-formula is a Π_n-formula, and vice versa; and that the formulas $\mathsf{Seq}(-)$, $\mathsf{Sent}(-)$ and $\mathsf{ConjSeq}(-)$ can be taken to be Σ_0, and the functions $\ell\mathsf{en}(-)$, $[-]_-$ and $\langle - | - \rangle$ are definable by Σ_0-formulas.

- The set of all true arithmetical formulas is denoted by $\mathrm{Th}(\mathbb{N})$; that is $\{\theta \in \mathrm{Sent} \mid \mathbb{N} \models \theta\}$. Similarly, for any n, Σ_n-$\mathrm{Th}(\mathbb{N}) = \{\theta \in \Sigma_n$-$\mathrm{Sent} \mid \mathbb{N} \models \theta\}$ and Π_n-$\mathrm{Th}(\mathbb{N}) = \{\theta \in \Pi_n$-$\mathrm{Sent} \mid \mathbb{N} \models \theta\}$. While by Tarski's Undefinability Theorem the (Gödel numbers of the members of the) set $\mathrm{Th}(\mathbb{N})$ is not definable, for $n > 0$ the (Gödel numbers of the members of the) set Σ_n-$\mathrm{Th}(\mathbb{N})$ is definable by the Σ_n-formula Σ_n-$\mathsf{True}(x)$ (stating that "x is the Gödel number of a true Σ_n-sentence") and the (Gödel numbers of the members of the) set Π_n-$\mathrm{Th}(\mathbb{N})$ is definable by the Π_n-formula Π_n-$\mathsf{True}(x)$ (stating that "x is the Gödel number of a true Π_n-sentence"). Robinson's Arithmetic is denoted by Q which is a weak (induction-free) fragment of PA.

- A definable theory is the set of all logical consequences of a set of sentences that (the set of the Gödel numbers of its members) is definable by an arithmetical formula $\mathsf{Axioms}_T(x)$ [meaning that x is the Gödel number of an axiom of T]. The formula $\mathsf{ConjAx}_T(x)$ states that "x is the Gödel code of a formula which is a conjunction of some axioms of T", i.e., $x = \ulcorner \bigwedge_{i=1}^{\ell} \varphi_i \urcorner$ where $\bigwedge_{i=1}^{\ell} \mathsf{Axioms}_T(\ulcorner \varphi_i \urcorner)$. The proof predicate of first order logic is denoted by $\mathsf{Proof}(y, x)$ which is a Σ_0-formula stating that "y is the code of a proof of the formula with code x in the first order logic". So, for a definable theory T the provability predicate of T is the formula $\mathsf{Prov}_T(x) = \exists y, z [\mathsf{ConjAx}_T(z) \wedge \mathsf{Proof}(y, z \to x)]$; also the consistency predicate of T is $\mathsf{Con}(T) = \neg\mathsf{Prov}_T(\ulcorner 0 \neq 0 \urcorner)$. Let us note that Prov_T defines the set of T-provable formulas, the deductive closure of (the axioms of) T. For a class of formulas Γ the theory T is called Γ-definable when $\mathsf{Axioms}_T \in \Gamma$. Let us also note that if $\mathsf{Axioms}_T \in \Sigma_{n+1}$ or $\mathsf{Axioms}_T \in \Pi_n$ then $\mathsf{ConjAx}_T \in \Sigma_{n+1}$ or $\mathsf{ConjAx}_T \in \Pi_n$, respectively, and so in either case $\mathsf{Prov}_T \in \Sigma_{n+1}$.

- Theory T decides the sentence φ when either $T \vdash \varphi$ or $T \vdash \neg\varphi$. A theory is called complete when it can decide every sentence in its language. A theory T is called Γ-deciding when it can decide any sentence in Γ. In the literature, a theory T is called Γ-complete when for any sentence $\varphi \in \Gamma$, if $\mathbb{N} \models \varphi$ then $T \vdash \varphi$. Note that if a sound theory is Γ-deciding then it is Γ-complete. A theory T is called ω-consistent when for no formula φ both the conditions (i) $T \vdash \neg\varphi(\overline{n})$ for all $n \in \mathbb{N}$, and (ii) $T \vdash \exists x \varphi(x)$ hold together. It is called n-consistent when for no formula $\varphi \in \Sigma_n$ with $\varphi = \exists x \psi(x)$ and $\psi \in \Pi_{n-1}$ one has (i) $T \vdash \neg\psi(\overline{n})$ for all $n \in \mathbb{N}$, and (ii) $T \vdash \varphi$. Theory T is called Γ-Sound, when for any sentence $\varphi \in \Gamma$, if $T \vdash \varphi$ then $\mathbb{N} \models \varphi$. For example, any consistent theory containing Π_n-$\mathrm{Th}(\mathbb{N})$ is Σ_n-sound. Let us note that, since $\mathrm{Th}(\mathbb{N})$ is a complete and thus a maximally consistent theory, the soundness of T is equivalent to $\mathrm{Th}(\mathbb{N}) \subseteq T$ and to the consistency of $T + \mathrm{Th}(\mathbb{N})$. In general, for any consistent extension T of Q, the Σ_n-soundness of T is equivalent to the consistency of $T + \Pi_n$-$\mathrm{Th}(\mathbb{N})$ (cf. Theorems 26,31 of [3]). Also, for any $T \supseteq Q$, since Q is a Σ_1-complete theory, the consistency of T is equivalent to the consistency of $T + \Pi_0$-$\mathrm{Th}(\mathbb{N})$, i.e. $\mathsf{Con}(T + \Pi_0$-$\mathrm{Th}(\mathbb{N}))$, which, in turn, is equivalent to the Σ_0-soundness of T (cf. Theorem 5 of [3]).

Semantic Condition		Conventional Notation		Syntactic Condition
(Σ_∞)Soundness of T	$\equiv$	$\mathbb{N} \models T$	$\equiv$	$\mathsf{Con}(T + [\Pi_\infty]\mathrm{Th}(\mathbb{N}))$
Σ_n-Soundness of T	$\equiv$	———	$\equiv$	$\mathsf{Con}(T + \Pi_n$-$\mathrm{Th}(\mathbb{N}))$
Σ_1-Soundness of T	$\equiv$	1-$\mathsf{Con}(T)$	$\equiv$	$\mathsf{Con}(T + \Pi_1$-$\mathrm{Th}(\mathbb{N}))$
Σ_0-Soundness of T	$\equiv$	$\mathsf{Con}(T)$	$\equiv$	$\mathsf{Con}(T + \Pi_0$-$\mathrm{Th}(\mathbb{N}))$

1.2　Some Earlier Attempts and Results

By Gödel's incompleteness theorem, PA (and every RE extension of it) is not Π_1-complete; then what about $\mathbf{S} = PA + \Pi_1\text{-Th}(\mathbb{N})$? Is this theory complete? For sure, it is Π_1-complete and Σ_1-complete; but can it be, say, Π_2-complete? Let us note that $\mathbf{S}$ is a Π_1-definable theory; i.e. $\mathsf{Axioms}_{\mathbf{S}} \in \Pi_1$, and so $\mathsf{Prov}_{\mathbf{S}} \in \Sigma_2$. So, it is natural to ask if the incompleteness phenomena still hold for definable arithmetical theories.

1.2.1　Results of Jeroslow (1975)

Jeroslow [5] showed in 1975 that when the set of theorems of a consistent theory that contains PA is Δ_2-definable, then it cannot contain the set of all true Π_1-sentences.

$$\boxed{\text{Jeroslow (1975)}: \qquad PA \subseteq T \ \& \ \mathsf{Prov}_T \in \Delta_2 \ \& \ \mathsf{Con}(T) \quad \Longrightarrow \quad \Pi_1\text{-Th}(\mathbb{N}) \nsubseteq T}$$

This result casts a new light on a classical theorem on the existence of a Δ_2-definable complete extension of PA (see [14]): no such complete extension can contain all the true Π_1-sentences. Note that one cannot weaken the assumption $\mathsf{Prov}_T \in \Delta_2$ in the theorem, to, say, $\mathsf{Prov}_T \in \Sigma_2$ because e.g. for the theory $\mathbf{S}$ above we have $\mathsf{Prov}_{\mathbf{S}} \in \Sigma_2$ and $\Pi_1\text{-Th}(\mathbb{N}) \subseteq \mathbf{S}$.

1.2.2　Results of Hájek (1977)

Jeroslow's theorem was generalized by Hájek ([1]) who showed that when the set of theorems of a consistent theory that contains PA is Δ_n-definable, then it cannot be Π_{n-1}-complete:

$$\boxed{\text{Hájek (1977)}: \qquad PA \subseteq T \ \& \ \mathsf{Prov}_T \in \Delta_n \ \& \ \mathsf{Con}(T) \quad \Longrightarrow \quad \Pi_{n-1}\text{-Th}(\mathbb{N}) \nsubseteq T}$$

Another result of Hájek ([1]) is that if a deductively closed extension of PA is Π_n-definable and n-consistent, then it cannot be Π_{n-1}-complete:

$$\boxed{\text{Hájek (1977a)}: \qquad PA \subseteq T \ \& \ \mathsf{Prov}_T \in \Pi_n \ \& \ n\text{–}\mathsf{Con}(T) \quad \Longrightarrow \quad \Pi_{n-1}\text{-Th}(\mathbb{N}) \nsubseteq T}$$

He also showed that no such theory can be complete; i.e., when $PA \subseteq T \ \& \ \mathsf{Prov}_T \in \Pi_n \ \& \ n\text{–}\mathsf{Con}(T)$ then T is incomplete (indeed, a Π_n-sentence is independent from T). Here, we generalize this theorem by showing the existence of an independent Π_{n-1}-sentence:

$$\boxed{\textbf{Corollary 1.3}: \qquad PA \subseteq T \ \& \ \mathsf{Prov}_T \in \Pi_n \ \& \ n\text{–}\mathsf{Con}(T) \quad \Longrightarrow \quad T \notin \Pi_{n-1}\text{-Deciding}}$$

Remark 1.1 (On the Proof of Theorem 2.5 in [1]) In Theorem 2.5 of [1] an n-consistent theory T is assumed to contain Peano Arithmetic (and be closed under deduction) and its set of theorems is assumed to be Π_n-definable for some $n \geqslant 2$. Then it is shown that (1) $\Pi_{n-1}\text{-Th}(\mathbb{N}) \nsubseteq T$, and a proof is presented for the fact that (2) T is incomplete.
In the proof of (1) for the sake of contradiction it is assumed that $\Pi_{n-2}\text{-Th}(\mathbb{N}) \subseteq T$; and at the end of the proof of (2) the inconsistency of T has been inferred from the T-provability of a false Π_{n-2}-sentence (denoted by $\tau_1(\overline{p}, \overline{m}, \overline{\varphi})$ in [1]). Of course, when $\Pi_{n-2}\text{-Th}(\mathbb{N}) \subseteq T$ then no false Π_{n-2}-sentence is provable in T. Probably, the proof did not intend to make use of the (wrong) assumption (of $\Pi_{n-2}\text{-Th}(\mathbb{N}) \subseteq T$); rather the intention could have been using the completeness and n-consistency of T to show that T cannot prove any false Π_{n-2}-sentence. This is the subject of the next lemma (1.2) which fills an inessential minor gap in the proof of [1, Theorem 2.5]. ✧

The following lemma generalizes Theorem 20 of [3] which states that *the true arithmetic* Th($\mathbb{N}$) *is the only ω-consistent extension of* PA (*indeed* Q) *that is complete*.

Lemma 1.2 (A Gap in the Proof of Theorem 2.5(2) in [1]) *Any n-consistent and Π_n-deciding extension of* Q *is Π_n-complete.*

Proof. By induction on n. For $n = 0$ there is nothing to prove. If the theorem holds for n then we prove it for $n + 1$ as follows. If T is $(n + 1)$-consistent and Π_{n+1}-deciding, but not Π_{n+1}-complete, there must exist some $\psi \in \Pi_{n+1}$-Th($\mathbb{N}$) such that $T \nvdash \psi$. Write $\psi = \forall z \eta(z)$ for some $\eta \in \Sigma_n$; then $\mathbb{N} \models \eta(m)$ for any $m \in \mathbb{N}$. By the induction hypothesis, T is Π_n-complete and so Σ_n-complete; thus $T \vdash \eta(\overline{m})$ for all $m \in \mathbb{N}$. On the other hand since T is Π_{n+1}-deciding and $T \nvdash \psi$ we must have $T \vdash \neg\psi$, thus $T \vdash \exists z \neg\eta(z)$. This contradicts the $(n + 1)$-consistency of T. ❑

Corollary 1.3 (Generalizing Theorem 2.5(2) of [1]) *If the deductive closure of an n-consistent extension of* PA *is Π_n-definable, then it has an independent Π_{n-1}-sentence (for any $n \geqslant 2$).*

Proof. If for a theory T we have PA $\subseteq T$ and $\mathsf{Prov}_T \in \Pi_n$ and n–Con(T) then it cannot be Π_{n-1}-deciding, since otherwise by Lemma 1.2 (and $(n-1)$-consistency of T), Π_{n-1}-Th($\mathbb{N}$) $\subseteq T$; this is in contradiction with Theorem 2.5(1) of [1] which states that Π_{n-1}-Th($\mathbb{N}$) $\not\subseteq T$ under the above assumptions. ❑

Below we will give yet another generalization of the above corollary (and a result of [1]) in Corollary 2.6. Hájek [1] has also showed that if the set of axioms of a consistent theory is Π_1-definable and that theory contains PA and all the true Π_1-sentences, then it is not Π_2-deciding. In Corollary 2.5 we will generalize this result by showing that no consistent Π_n-definable and Π_n-complete extension of Q is Π_{n+1}-deciding.

Corollary 2.5 : $\quad$ Q $\subseteq T$ & $\mathsf{Axioms}_T \in \Pi_n$ & Con(T) & Π_n-Th($\mathbb{N}$) $\subseteq T$ $\implies$ $T \notin \Pi_{n+1}$-Deciding

1.2.3 Some Recent Attempts

In the result of Jeroslow (Theorem 2 of [5]) and Hájek's generalizations (Theorem 2.5(1) and Theorem 2.8 of [1]) there is no incompleteness; we have only some non-inclusion (of the set of true Π_1 or Π_n sentences in the theory). In the incompleteness theorems of Hájek ([1] and Corollaries 1.3 and 2.5) we had the, somewhat strong, assumptions of n-consistency or Π_n-completeness (with consistency). It is natural to ask if we can weaken these assumptions (like in Rosser's Trick) to mere consistency; and some attempts [7, 4] have been made in this direction. Let us note that Rosserian (also Gödelean) proofs make sense for definable theories only (for example the undefinable theory Th($\mathbb{N}$) is complete) for the reason that when a theory T is definable one can construct its provability predicate Prov_T, and once one has a provability predicate for T then it becomes a definable theory.

We note that the proofs of Gödel-Rosser's incompleteness theorem for non-RE theories given in [7, 4] are both wrong; for the falsity of the argument of [7] one can see [10]; cf. also [8] and [11]. The falsity of the proof of [4] is shown in the following remark (1.4). Unfortunately, there is no hope of extending Gödel-Rosser's incompleteness theorem to definable theories, even to Π_1-definable ones; our Corollary 3.4 below shows that even a (consistent and) Π_1-definable theory (extending Q) can be complete. This clashes all the hopes for a general incompleteness phenomenon in the class of definable, and consistent, theories.

Remark 1.4 Unfortunately, the proof of Gödel-Rosser's incompleteness theorem for non-RE theories given in [4] is wrong: In the proof of Lemma 3 in [4] the author uses the Diagonal Lemma for $\neg F(x)$, where F is constructed in Lemma S (Chapter VI) of [15] (together with its Lemma 2 in Chapter V); it can be seen that $F \in \Pi_1$ and so $A \in \Sigma_1$. If, as claimed in Lemma 3 (and Theorem and Corollary) of [4], for any

Π_1-definable consistent extension of Q there existed a Σ_1-sentence A independent from it, then the theory $Q + \Pi_1\text{-Th}(\mathbb{N})$ would have had a Π_1-sentence independent from it. But it is well-known that this theory is Σ_1-complete and Π_1-complete. So, the proof of the main theorem of [4] is flawed. In fact, the mistaken step is in the proof of Lemma 2 where the author claims that "m_1 can be chosen such that $m_2 \leqslant m_1$ and hence $\overline{R}(k, m_2, \text{Neg}(n))$." But if we choose m_1 arbitrarily large then the condition $\forall x \leqslant m_1 \overline{R}(k, x, \text{Neg}(n))$ may not necessarily hold anymore. Indeed, Theorem 3.1 for $n = 0$ is the negation of what is claimed in the Abstract of [4]. ✧

2 Gödel's Theorem Generalized

2.1 Semantic Form of Gödel's Theorem

Gödel's First Incompleteness Theorem in its (weaker) semantic form states that no sound and RE extension of Q can be Π_1-complete. Noting that a set is RE if and only if it is Σ_1-definable, this theorem can be depicted as:

Gödel's 1$^{\text{st}}$ (Semantic) : $Q \subseteq T$ & $\text{Axioms}_T \in \Sigma_1$ & $\mathbb{N} \models T$ $\implies$ $\Pi_1\text{-Th}(\mathbb{N}) \nsubseteq T$

A natural generalization of this theorem is the following (cf. Chapter III of [15], or Corollary 1 of [12]):

Theorem 2.1 *No sound and Σ_n-definable ($n > 0$) extension of Q can be Π_n-complete.*

Theorem 2.1 : $Q \subseteq T$ & $\text{Axioms}_T \in \Sigma_n$ & $\mathbb{N} \models T$ $\implies$ $\Pi_n\text{-Th}(\mathbb{N}) \nsubseteq T$

Proof. Suppose T is a sound extension of Q such that $\text{Axioms}_T \in \Sigma_n$. By Diagonal Lemma (see e.g. [2, 13]) there exists a sentence $\boldsymbol{\gamma}$ such that $Q \vdash \boldsymbol{\gamma} \longleftrightarrow \neg\text{Prov}_T(\ulcorner\boldsymbol{\gamma}\urcorner)$. Obviously, $\boldsymbol{\gamma} \in \Pi_n$. We show that (†) $\mathbb{N} \models \boldsymbol{\gamma}$.

Since, otherwise (if $\mathbb{N} \models \neg\boldsymbol{\gamma}$ then) there must exist some $k, m \in \mathbb{N}$ such that $\mathbb{N} \models \text{ConjAx}_T(k)$ and $\mathbb{N} \models \text{Proof}(m, k \to \ulcorner\boldsymbol{\gamma}\urcorner)$. Whence, $T \vdash \boldsymbol{\gamma}$ which contradicts the soundness of T. So, $\mathbb{N} \models \boldsymbol{\gamma}$. Now, we show that $T \nvdash \boldsymbol{\gamma}$. For the sake of contradiction, assume $T \vdash \boldsymbol{\gamma}$. Then, by the compactness theorem, there are some $\varphi_1, \cdots, \varphi_l$ such that $\mathbb{N} \models \bigwedge_{i=1}^{l} \text{Axioms}_T(\ulcorner\varphi_i\urcorner)$ and $\vdash \bigwedge_{i=1}^{l} \varphi_i \to \boldsymbol{\gamma}$. If m is the code of this proof and k is the code of $\bigwedge_{i=1}^{l} \varphi_i$ then $\mathbb{N} \models \text{ConjAx}_T(k) \wedge \text{Proof}(m, k \to \ulcorner\boldsymbol{\gamma}\urcorner)$, or in other words $\mathbb{N} \models \text{Prov}_T(\ulcorner\boldsymbol{\gamma}\urcorner)$ so $\mathbb{N} \models \neg\boldsymbol{\gamma}$ contradicting (†) above. Thus, $\boldsymbol{\gamma} \in \Pi_n\text{-Th}(\mathbb{N}) \setminus T$. ❏

Corollary 2.2 *No sound and Π_n-definable extension of Q can be Π_{n+1}-complete.*

Corollary 2.2 : $Q \subseteq T$ & $\text{Axioms}_T \in \Pi_n$ & $\mathbb{N} \models T$ $\implies$ $\Pi_{n+1}\text{-Th}(\mathbb{N}) \nsubseteq T$

Proof. It suffices to note that any Π_n-definable is also Σ_{n+1}-definable. ❏

Remark 2.3 It is well known that Q is Σ_1-complete (see e.g. [2, 6, 13]) but not Π_1-complete (by Gödel's first incompleteness theorem, see e.g. [2, 6, 13]). So, Σ_1-completeness does not imply Π_1-completeness, and in general, Σ_n-completeness does not imply Π_n-completeness, since for example the Σ_n-complete and sound theory $Q + \Sigma_n\text{-Th}(\mathbb{N})$ is not Π_n-complete by Theorem 2.1. On the other hand, Π_n-completeness (of any theory T) implies (its) Σ_n-completeness, even (its) Σ_{n+1}-completeness: for any true Σ_{n+1}-sentence $\exists x_1, \ldots, x_k \theta(x_1, \ldots, x_k)$ with $\theta \in \Pi_n$ there are $n_1, \ldots, n_k \in \mathbb{N}$ such that $\mathbb{N} \models \theta(n_1, \ldots, n_k)$, and so by Π_n-completeness of T we have $T \vdash \theta(\overline{n_1}, \ldots, \overline{n_k})$ whence $T \vdash \exists x_1, \ldots, x_k \theta(x_1, \ldots, x_k)$. In symbols: $\Pi_n\text{-Th}(\mathbb{N}) \subseteq T \implies \Sigma_{n+1}\text{-Th}(\mathbb{N}) \subseteq T$ (cf. [1, Lemma 2.2]). ✧

2.2 General Form of Gödel's Theorem

The original form of Gödel's first incompleteness theorem states that a recursively enumerable extension of Q which is ω-consistent cannot be Π_1-deciding. This syntactic notion was introduced to take place of the semantic notion of soundness. Later it was found out that Gödel's proof works with the weaker assumption of 1-consistency which is equivalent to the consistency (of the theory) with Π_1-Th($\mathbb{N}$) (see [3]):

$$\boxed{\text{Gödel's 1}^{\text{st}} \text{ (1931)}: \qquad Q \subseteq T \ \& \ \mathsf{Axioms}_T \in \Sigma_1 \ \& \ \mathsf{Con}\big(T + \Pi_1\text{-Th}(\mathbb{N})\big) \implies T \notin \Pi_1\text{-Deciding}}$$

A natural generalization of the theorem in this form is the Π_n-undecidability of any Σ_n-definable extension of Q which is consistent with Π_n-Th($\mathbb{N}$); proved in Corollary 2.8 of the following theorem.

Theorem 2.4 *No Π_n-definable extension of Q can be Π_{n+1}-deciding if it is consistent with Π_n-Th($\mathbb{N}$).*

$$\boxed{\textbf{Theorem 2.4}: \qquad Q \subseteq T \ \& \ \mathsf{Axioms}_T \in \Pi_n \ \& \ \mathsf{Con}\big(T + \Pi_n\text{-Th}(\mathbb{N})\big) \implies T \notin \Pi_{n+1}\text{-Deciding}}$$

Proof. By Diagonal Lemma there exists a sentence $\boldsymbol{\gamma}$ such that

$$Q \vdash \boldsymbol{\gamma} \longleftrightarrow \forall u, z \Big(\exists x, y \leqslant u \big[\langle x, y \rangle = u \wedge \Pi_n\text{-True}(x) \wedge \mathsf{ConjAx}_T(y) \wedge \mathsf{Proof}(z, x \wedge y \to \ulcorner\boldsymbol{\gamma}\urcorner) \big] \to$$

$$\exists u' \leqslant u \exists z' \leqslant z \big(\exists x', y' \leqslant u' \big[\langle x', y' \rangle = u' \wedge \Pi_n\text{-True}(x') \wedge \mathsf{ConjAx}_T(y') \wedge \mathsf{Proof}(z', x' \wedge y' \to \ulcorner\neg\boldsymbol{\gamma}\urcorner) \big] \big) \Big) \qquad (\star)$$

where, $\langle -, - \rangle$ is an injective pairing (such as $\langle u, v \rangle = (u + v)^2 + u$).

Obviously, $\boldsymbol{\gamma} \in \Pi_{n+1}$. We show that $\boldsymbol{\gamma}$ is independent from $T^* = T + \Pi_n\text{-Th}(\mathbb{N})$.

Put $\Psi(u, z) = \exists x, y \leqslant u \big[\langle x, y \rangle = u \wedge \Pi_n\text{-True}(x) \wedge \mathsf{ConjAx}_T(y) \wedge \mathsf{Proof}(z, x \wedge y \to \ulcorner\boldsymbol{\gamma}\urcorner) \big]$ and

$\widehat{\Psi}(u, z) = \exists x, y \leqslant u \big[\langle x, y \rangle = u \wedge \Pi_n\text{-True}(x) \wedge \mathsf{ConjAx}_T(y) \wedge \mathsf{Proof}(z, x \wedge y \to \ulcorner\neg\boldsymbol{\gamma}\urcorner) \big]$.

Thus, $(\star)$ is now translated to

$$Q \vdash \boldsymbol{\gamma} \longleftrightarrow \forall u, z \big[\Psi(u, z) \to \exists u' \leqslant u \exists z' \leqslant z \widehat{\Psi}(u', z') \big]. \tag{$\star'$}$$

$(T^* \nvdash \boldsymbol{\gamma})$:   If $T^* \vdash \boldsymbol{\gamma}$ then there are $\psi \in \Pi_n\text{-Th}(\mathbb{N})$ (note that $\Pi_n\text{-Th}(\mathbb{N})$ is closed under conjunction) and a conjunction φ of the axioms of T such that $\vdash \psi \wedge \varphi \to \boldsymbol{\gamma}$. Let m be the Gödel code of this proof and let $k = \langle \ulcorner\psi\urcorner, \ulcorner\varphi\urcorner \rangle$. Now, we have $\mathbb{N} \models \Psi(k, m)$, and so by the Π_n-completeness of T^* we have $T^* \vdash \Psi(\overline{k}, \overline{m})$, thus by $(\star')$, $T^* \vdash \exists u' \leqslant \overline{k} \exists z' \leqslant \overline{m} \widehat{\Psi}(u', z')$.   ($\ddagger$)

On the other hand by the consistency of T^* we have $T^* \nvdash \neg\boldsymbol{\gamma}$. So, for any $q = \langle q_1, q_2 \rangle, r \in \mathbb{N}$ we have that if $\mathbb{N} \models \Pi_n\text{-True}(q_1) \wedge \mathsf{ConjAx}_T(q_2)$ then $\mathbb{N} \models \neg\mathsf{Proof}(r, q_1 \wedge q_2 \to \ulcorner\neg\boldsymbol{\gamma}\urcorner)$. Whence, $\mathbb{N} \models \neg\widehat{\Psi}(q, r)$ holds for all $q, r \in \mathbb{N}$ in particular for all $q \leqslant k, r \leqslant m$; thus $\mathbb{N} \models \forall u' \leqslant k \forall z' \leqslant m \neg\widehat{\Psi}(u', z')$. Now, $\forall u' \leqslant k \forall z' \leqslant m \neg\widehat{\Psi}(u', z')$ is a true Σ_n-sentence and T^* is a Π_n-complete theory; so by Remark 2.3, $T^* \vdash \forall u' \leqslant \overline{k} \forall z' \leqslant \overline{m} \neg\widehat{\Psi}(u', z')$ contradicting ($\ddagger$) above!

$(T^* \nvdash \neg\boldsymbol{\gamma})$: If $T^* \vdash \neg\boldsymbol{\gamma}$ then from $(\star')$ it follows that

$$(i) \qquad T^* \vdash \exists u, z \big[\Psi(u, z) \wedge \forall u' \leqslant u \forall z' \leqslant z \neg\widehat{\Psi}(u', z') \big].$$

By the compactness theorem (applied to the deduction $T^* \vdash \neg\boldsymbol{\gamma}$) there are $k = \langle k_1, k_2 \rangle, m \in \mathbb{N}$ such that $\mathbb{N} \models \Pi_n\text{-True}(k_1) \wedge \mathsf{ConjAx}_T(k_2) \wedge \mathsf{Proof}(m, k_1 \wedge k_2 \to \ulcorner\neg\boldsymbol{\gamma}\urcorner)$. Below, we will show that

$$(ii) \qquad T^* \vdash \forall u, z \big[\neg\Psi(u, z) \vee \exists u' \leqslant u \exists z' \leqslant z \widehat{\Psi}(u', z') \big],$$

which contradicts (i) above. The proof of (ii) will be in three steps:

$$(1) \quad T^* \vdash \forall u \geqslant \overline{k} \forall z \geqslant \overline{m} \big[\exists u' \leqslant u \exists z' \leqslant z \widehat{\Psi}(u', z') \big]$$

$$(2) \quad T^* \vdash \forall u < \overline{k} \forall z \big[\neg\Psi(u, z) \big]$$

$$(3) \quad T^* \vdash \forall u \forall z < \overline{m} \big[\neg\Psi(u, z) \big]$$

(1) Since $\widehat{\Psi}(\overline{k}, \overline{m}) = \exists x, y \leqslant \overline{k} \big[\langle x, y \rangle = \overline{k} \wedge \Pi_n\text{-True}(x) \wedge \mathsf{ConjAx}_T(y) \wedge \mathsf{Proof}(\overline{m}, x \wedge y \to \ulcorner\neg\boldsymbol{\gamma}\urcorner) \big]$ is a true Π_n-sentence (for $x = k_1, y = k_2$), then T^* proves it, so (1) holds (for $u' = \overline{k}, z' = \overline{m}$).

(2) It suffices to show $T^* \vdash \forall z \neg \Psi(\bar{i}, z)$ for all $i < k$. Fix an $i < k$. If there are no i_1, i_2 such that $\langle i_1, i_2 \rangle = i$ then $T^* \vdash \forall z \neg \Psi(\bar{i}, z)$ holds trivially; otherwise, fix i_1, i_2 with $\langle i_1, i_2 \rangle = i$. If either $\neg \Pi_n\text{-True}(i_1)$ or $\neg \mathsf{ConjAx}_T(i_2)$, then again $T^* \vdash \forall z \neg \Psi(\bar{i}, z)$ holds. Finally, assume that $\Pi_n\text{-True}(i_1) \wedge \mathsf{ConjAx}_T(i_2)$ is true. Then, by the consistency of T^* we have $T^* \not\vdash \boldsymbol{\gamma}$ and so for all $p \in \mathbb{N}$ we have $\mathbb{N} \models \neg \mathsf{Proof}(p, i_1 \wedge i_2 \to \ulcorner \boldsymbol{\gamma} \urcorner)$. Whence, T^* proves the true Π_1-sentence $\forall z \neg \mathsf{Proof}(z, i_1 \wedge i_2 \to \ulcorner \boldsymbol{\gamma} \urcorner)$, and so $T^* \vdash \forall z \neg \Psi(\bar{i}, z)$.

(3) Again we need to show $T^* \vdash \forall u \big[\neg \Psi(u, \bar{j}) \big]$ for all $j < m$. Since T^* proves the true Π_1-sentence $\forall x, y, v, w \big[\mathsf{Proof}(w, x \wedge y \to v) \to \langle x, y \rangle < w \big]$ then $T^* \vdash \forall u \big[\Psi(u, \bar{j}) \to u < \bar{j} \big]$. Since, by an argument similar to that of (2) above, we can show that $T^* \vdash \forall u < \bar{j} \big[\neg \Psi(u, \bar{j}) \big]$, then $T^* \vdash \forall u \forall z < \overline{m} \big[\neg \Psi(u, z) \big]$ holds too.

Whence, T^*, and so T, is not Π_{n+1}-deciding. Let us note that the above proof also shows that $\mathbb{N} \models \boldsymbol{\gamma}$. ❑

Note that Theorem 2.4 is Rosser's Theorem for $n = 0$, and indeed one can feel that the above, rather long, proof is in spirit more Rosserian (than Gödelean) in the sense that the proof uses somehow Rosser's Trick.

Corollary 2.5 *No consistent Π_n-definable and Π_n-complete extension of Q can be Π_{n+1}-deciding.*

Proof. If $T \supseteq Q + \Pi_n\text{-Th}(\mathbb{N})$ is consistent and Π_n-definable, then $T + \Pi_n\text{-Th}(\mathbb{N})$ is consistent, and so by Theorem 2.4, T is not Π_{n+1}-deciding. ❑

Corollary 2.6 *No Π_n-definable extension of Q can be Π_{n+1}-deciding if it is n-consistent.*

$$\boxed{\textbf{Corollary 2.6}: \qquad Q \subseteq T \ \& \ \mathsf{Axioms}_T \in \Pi_n \ \& \ n\text{-}\mathsf{Con}(T) \implies T \notin \Pi_{n+1}\text{-Deciding}}$$

Proof. Let $T \supseteq Q$ be an n-consistent extension of Q such that $\mathsf{Axioms}_T \in \Pi_n$. If T is not Π_n-deciding, then there is nothing to prove. If T is Π_n-deciding, then by Lemma 1.2 we have $\Pi_n\text{-Th}(\mathbb{N}) \subseteq T$, and so T is consistent with $\Pi_n\text{-Th}(\mathbb{N})$. Thus, by Theorem 2.4, T is not Π_{n+1}-deciding. ❑

Let us note that for a Π_n-definable extension of Q (like T) Corollary 1.3 implies the Π_{n+1}-undecidability (of T) under the condition of $(n+2)$-consistency (of T) because $\mathsf{Axioms}_T \in \Pi_n$ implies $\mathsf{Prov}_T \in \Pi_{n+2}$; while Corollary 2.6 derives the same conclusion (of the Π_{n+1}-undecidability of T) under the assumption of n-consistency (of T). So, we can argue that Theorem 2.4 somehow strengthens Theorem 2.5(2) of [1]. The following lemma, needed later, generalizes (and modifies) Craig's Trick.

Lemma 2.7 *Any Σ_{n+1}-definable (arithmetical) theory is equivalent with a Π_n-definable theory.*

Proof. If $\mathsf{Axioms}_T(x) = \exists x_1 \cdots \exists x_n \theta(x, x_1, \cdots, x_n)$ with $\theta \in \Pi_n$ then $\mathsf{Axioms}_T(x) \equiv \exists y \theta'(x, y)$ with $\theta'(x, y) = \exists x_1 \leqslant y \cdots \exists x_n \leqslant y \, \theta(x, x_1, \cdots, x_n) \in \Pi_n$. Now, $T' = \{ \varphi \wedge (\overline{k} = \overline{k}) \mid \mathbb{N} \models \theta'(\ulcorner \varphi \urcorner, k) \}$ is equivalent with T and is Π_n-definable by $\mathsf{Axioms}_{T'}(x) \equiv \exists y, z \leqslant x \big(\theta'(y, z) \wedge \big[x = (y \wedge \ulcorner \overline{z} = \overline{z} \urcorner) \big] \big)$. ❑

Corollary 2.8 *No Σ_n-definable $(n > 0)$ extension of Q can be Π_n-deciding if it is consistent with Π_n-Th($\mathbb{N}$).*

$$\boxed{\textbf{Corollary 2.8}: \qquad Q \subseteq T \ \& \ \mathsf{Axioms}_T \in \Sigma_n \ \& \ \mathsf{Con}\big(T + \Pi_n\text{-Th}(\mathbb{N})\big) \implies T \notin \Pi_n\text{-Deciding}}$$

Proof. For $n = 1$ this is Gödel's first incompleteness theorem. Suppose that $n > 1$, and that $\mathsf{Axioms}_T \in \Sigma_n$ for some $T \supseteq Q$ such that $T + \Pi_n\text{-Th}(\mathbb{N})$ is consistent. By Lemma 2.7 there exists a Π_{n-1}-definable theory T' equivalent with T. Now, T' contains Q, is Π_{n-1}-definable, and is consistent with $\Pi_{n-1}\text{-Th}(\mathbb{N})$ (because T is consistent with $\Pi_n\text{-Th}(\mathbb{N})$). Thus, by Theorem 2.4 the theory T' is not Π_n-deciding; neither is T. ❑

Actually, the consistency of T with $\Pi_{n-1}\text{-Th}(\mathbb{N})$ suffices for the above proof to go through.

Corollary 2.9 *No Σ_n-definable extension of Q can be Π_n-deciding if it is consistent with Π_{n-1}-Th($\mathbb{N}$).* ❑

> **Corollary 2.9** : $Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & $\mathsf{Con}\big(T + \Pi_{n-1}\text{-Th}(\mathbb{N})\big) \implies T \notin \Pi_n\text{-Deciding}$

By Gödel's first incompleteness theorem no 1-consistent and Σ_1-definable extension of Q can be Π_1-deciding; another generalization of this theorem is the Π_n-undecidability of any n-consistent and Σ_n-definable extension of Q.

Corollary 2.10 *No Σ_n-definable extension of Q can be Π_n-deciding if it is n-consistent.*

> **Corollary 2.10** : $Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & $n\text{-}\mathsf{Con}(T) \implies T \notin \Pi_n\text{-Deciding}$

Proof. By Lemma 2.7 any Σ_n-definable theory is equivalent with a Π_{n-1}-definable theory, and if that theory is $(n-1)$-consistent, then (extending Q) it cannot be Π_n-deciding by Corollary 2.6. ❑

In fact, we can prove even a more general theorem here: no $(n-1)$-consistent and Σ_n-definable extension of Q can be Π_n-deciding (because what was used in the above proof was the $(n-1)$-consistency of the theory); this is actually a generalization of Gödel-Rosser's incompleteness theorem.

Corollary 2.11 *No Σ_n-definable extension of Q can be Π_n-deciding if it is $(n-1)$-consistent.*   ❑

> **Corollary 2.11** : $Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & $(n-1)\text{-}\mathsf{Con}(T) \implies T \notin \Pi_n\text{-Deciding}$

3   Rosser's Theorem Optimized

Rosser's Trick is one of the most fruitful tricks in Mathematical Logic and Recursion Theory (cf. [15]). One of its uses is getting rid of the condition of ω-consistency (or 1-consistency or equivalently consistency with the set of true Π_1-sentences) from the hypothesis of Gödel's first incompleteness theorem. Thus, Gödel-Rosser's incompleteness theorem (see e.g. [2, 13, 15]) can be depicted as:

> Gödel–Rosser (1936) : $Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_1$ & $\mathsf{Con}(T) \implies T \notin \Pi_1\text{-Deciding}$

In the light of our above mentioned results it is natural to expect a generalization of this theorem to higher levels (to Σ_n or Π_n definable theories); alas (by the following theorem for $n = 0$) there can bo no such generalization for Rosser's Theorem.

Theorem 3.1 *There exists a complete (and consistent) and Σ_{n+2}-definable extension of $Q + \Pi_n$-Th($\mathbb{N}$).*

Proof. That there exists a complete Σ_2-definable extension of Q is almost a classical fact; see [14]. Here, we generalize this result to $Q + \Pi_n$-Th($\mathbb{N}$). Let the theory S be Q when $n = 0$ and be $Q + \Pi_n$-Th($\mathbb{N}$) when $n > 0$ (note that Π_0-Th($\mathbb{N}$) $\subseteq Q$). Theory S can be completed by Lindenbaum's Lemma as follows: for an enumeration of all the sentences $\varphi_0, \varphi_1, \varphi_2, \cdots$ take $T_0 = S$, and let $T_{n+1} = T_n + \varphi_n$ if $\mathsf{Con}(T_n + \varphi_n)$ and let $T_{n+1} = T_n + \neg\varphi$ otherwise [if $\neg\mathsf{Con}(T_n + \varphi_n)$]. Then the theory $T^* = \bigcup_{n \in \mathbb{N}} T_n$ is a complete extension of S; below we show the Σ_{n+2}-definability of T^*. An enumeration of all the sentences can be defined by a Σ_0-formula such as the following expression for "x is the (Gödel number of the) u^{th} sentence":

$$\mathsf{Sent\text{-}List}(x, u) = \big[\mathsf{Sent}(u) \wedge x = u\big] \vee \big[\neg\mathsf{Sent}(u) \wedge x = \ulcorner 0 = 0 \urcorner\big].$$

Now, $\mathsf{Axioms}_{T^*}(x)$ can be defined by the following formula:

$$\exists y \Big[\mathsf{Seq}(y) \wedge [y]_{\ell\mathrm{en}(y)-1} = x \wedge \big(\forall u < \ell en(y) \big[\mathsf{Sent}([y]_u) \big] \big) \wedge$$

$$\forall u < \ell en(y) \forall z \leqslant y \Big(\big(\mathsf{Sent\text{-}List}(z,u) \wedge \mathsf{Con}'(S + \langle y \downarrow u \rangle + z) \longrightarrow [y]_u = z \big) \wedge$$

$$\big(\mathsf{Sent\text{-}List}(z,u) \wedge \neg\mathsf{Con}'(S + \langle y \downarrow u \rangle + z) \longrightarrow [y]_u = \neg z \big) \Big) \Big],$$

which is Σ_{n+2} because the following formula (where q is the Gödel code of the conjunction of all the [finitely many] axioms of Q and $\perp = [0 \neq 0]$)

$$\mathsf{Con}'(S + \langle y \downarrow u \rangle + z) \equiv \begin{cases} \forall v, w \big[\mathsf{ConjSeq}(v, \langle y \downarrow u \rangle) \to \neg\mathsf{Proof}(w, \mathsf{q} \wedge v \wedge z \to \ulcorner \perp \urcorner) \big] & \text{if } n = 0 \\ \forall t, v, w \big[\Pi_n\text{-}\mathsf{True}(t) \wedge \mathsf{ConjSeq}(v, \langle y \downarrow u \rangle) \to \neg\mathsf{Proof}(w, \mathsf{q} \wedge t \wedge v \wedge z \to \ulcorner \perp \urcorner) \big] & \text{if } n > 0 \end{cases}$$

is Π_{n+1} since $\Pi_n\text{-}\mathsf{True} \in \Pi_n$ (and $\mathsf{ConjSeq}, \mathsf{Proof} \in \Pi_0$). ❏

3.1 Comparing Σ_n-Soundness with n-Consistency

The assumptions on the theory T used in Corollaries 2.8 and 2.10, other than $Q \subseteq T \& \mathsf{Axioms}_T \in \Sigma_n$, are either consistency with the set of all true Π_n sentences (or equivalently, Σ_n-soundness) or n-consistency of T (cf. also Corollaries 2.9 and 2.11). So, it is desirable to compare the assumptions of Σ_n-soundness and n-consistency used in these results.

Proposition 3.2 (1) *If a theory is Σ_n-sound, then it is n-consistent.*
    (2) *If a Σ_{n-1}-complete theory is n-consistent, then it is Σ_n-sound.*

Proof. (1) Assume $T \vdash \exists x \psi(x)$ for some Σ_n-sound theory T and some formula $\psi \in \Pi_{n-1}$. By the Σ_n-soundness of T, $\mathbb{N} \models \exists x \psi(x)$, and so $\mathbb{N} \models \psi(m)$ for some $m \in \mathbb{N}$. Now, $\psi(\overline{m}) \in \Pi_{n-1}\text{-}\mathrm{Th}(\mathbb{N})$, and again by the Σ_n-soundness of T we have $T \nvdash \neg\psi(\overline{m})$.

    (2) Assume $T \vdash \exists x \psi(x)$ for some Σ_{n-1}-complete and n-consistent theory T and some formula $\psi \in \Pi_{n-1}$. By n-consistency, there exists some $m \in \mathbb{N}$ such that $T \nvdash \neg\psi(\overline{m})$. By Σ_{n-1}-completeness, $\mathbb{N} \nvDash \neg\psi(\overline{m})$; and so $\mathbb{N} \models \psi(\overline{m})$, whence $\mathbb{N} \models \exists x \psi(x)$. ❏

Remark 3.3 In fact, for $n = 0, 1, 2$ the notions of Σ_n-soundness and n-consistency are equivalent for Σ_1-complete theories (see Theorems 5,25,30 of [3]); but for $n \geqslant 3$, n-consistency does not imply Σ_n-soundness. Even, ω-consistency does not imply Σ_3-soundness (see Theorem 19 of [3] proved by Kreisel in 1955). Generally, Σ_n-soundness does not imply $(n+1)$-consistency: Let γ be the true Π_{n+1}-sentence constructed in Theorem 2.4 for the theory $Q + \Pi_n\text{-}\mathrm{Th}(\mathbb{N})$ and put $S = T + \neg\gamma$. Now, S is Σ_n-sound and not $(n+1)$-consistent, since for $\neg\gamma = \exists x \delta(x) \in \Sigma_{n+1}$ we have $S \vdash \exists x \delta(x)$ and for any $k \in \mathbb{N}$ we have $S \vdash \neg\delta(\overline{k})$ since S is Σ_{n+1}-complete by Remark 2.3 and $\neg\delta(\overline{k}) \in \Sigma_{n+1}\text{-}\mathrm{Th}(\mathbb{N})$ (because, if $\mathbb{N} \nvDash \neg\delta(\overline{k})$ then $\mathbb{N} \models \delta(\overline{k})$ and so $\mathbb{N} \models \neg\gamma$ contradiction!). ✧

$$\Sigma_0\text{-Sound} \Longleftarrow \Sigma_1\text{-Sound} \Longleftarrow \Sigma_2\text{-Sound} \Longleftarrow \Sigma_3\text{-Sound} \Longleftarrow \cdots \Sigma_n\text{-Sound} \cdots \Longleftarrow \text{Sound}$$
$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Downarrow \qquad\qquad\qquad \Downarrow \qquad \not\Uparrow \; \Downarrow$$
$$\text{Consistent} \Longleftarrow 1\text{-Consistent} \Longleftarrow 2\text{-Consistent} \Longleftarrow 3\text{-Consistent} \Longleftarrow \cdots n\text{-Consistent} \cdots \Longleftarrow \omega\text{-Consistent}$$

Corollary 3.4 (1) *There exists a complete extension of Q which is Σ_{n+2}-definable and consistent with $\Pi_n\text{-}\mathrm{Th}(\mathbb{N})$ (and so n-consistent).*

    (2) *There exists a complete extension of Q which is Π_{n+1}-definable and consistent with $\Pi_n\text{-}\mathrm{Th}(\mathbb{N})$ (and so n-consistent).*

Corollary 3.4(1) : $Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_{n+2}$ & $\big[\mathsf{Con}\big(T + \Pi_n\text{-}\mathrm{Th}(\mathbb{N})\big) \bigvee n\text{-}\mathsf{Con}(T)\big]$ $\not\Longrightarrow$ $T \notin \mathrm{Complete}$

Corollary 3.4(2) : $Q \subseteq T$ & $\mathsf{Axioms}_T \in \Pi_{n+1}$ & $\big[\mathsf{Con}\big(T + \Pi_n\text{-}\mathrm{Th}(\mathbb{N})\big) \bigvee n\text{-}\mathsf{Con}(T)\big]$ $\not\Longrightarrow$ $T \notin \mathrm{Complete}$

Proof. (1) The Σ_{n+2}-definable and complete extension of Q in Theorem 3.1 contains Π_n-$\mathrm{Th}(\mathbb{N})$, and so, being Σ_n-sound, is n-consistent by Proposition 3.2.

(2) The Σ_{n+2}-definable theory of part (1) is equivalent with a Π_{n+1}-definable theory by Lemma 2.7.   ❑

3.2   A Note on the Constructiveness of the Proofs

It is interesting to note that for $n \geqslant 3$ all the incompleteness proofs (presented here) with the assumption of Σ_n-soundness are constructive (i.e., the independent sentence can be effectively constructed from the given Σ_n-sound theory satisfying the conditions of Σ_n/Π_n definability), while all the incompleteness proofs (here) with the assumption of n-consistency are all non-constructive (i.e., the independent sentence is not constructed explicitly, and only its mere existence is proved). Our final result contains a bit of a surprise: even though the proof of Corollary 2.11 is not constructive, no one can present a constructive proof for it.

Theorem 3.5 (Non-Constructivity of n-Consistency Incompleteness) *Let $n \geqslant 3$ be fixed. There is no recursive function f (even with the oracle $\emptyset^{(n)}$) such that when m is a (Gödel code of a) Σ_{n+1}-formula which defines an n-consistent extension of Q, then $f(m)$ is a (Gödel code of a) Π_{n+1}-sentence independent from that theory.*

Proof. Assume that there is an $\emptyset^{(n)}$-recursive function f such that for any given Σ_{n+1}-formula $\Psi(x)$ if the theory $\mathcal{T}_\Psi = \{\alpha \mid \mathbb{N} \models \Psi(\ulcorner\alpha\urcorner)\}$ is an n-consistent extension of Q then $f(\ulcorner\Psi\urcorner)$ is (the Gödel code of) a Π_{n+1}-sentence such that $\mathcal{T}_\Psi \not\vdash f(\ulcorner\Psi\urcorner)$ and $\mathcal{T}_\Psi \not\vdash \neg f(\ulcorner\Psi\urcorner)$. The ω-consistency of Q with x can be written by the Π_3-formula $\boldsymbol{\omega}\text{-}\mathsf{Con}_Q(x) = \forall\chi\big[\exists z\mathsf{Proof}\big(z, \mathsf{q}\wedge x \to \exists v\chi(v)\big) \to \exists v\forall z\neg\mathsf{Proof}\big(z, \mathsf{q}\wedge x \to \neg\chi(\overline{v})\big)\big]$, where q is the Gödel code of the conjunction of the finitely many axioms of Q (see the Proof of Theorem 3.1). By $\emptyset^{(n)}$-recursiveness of f the expressions $y = f(x)$ and $f(z)\downarrow$ can be written by Σ_{n+1}-formulas (see e.g. [9]). By Diagonal Lemma there exists some Σ_{n+1}-formula $\Theta(x)$ such that

$$
\begin{aligned}
\Theta(x) \quad\equiv\quad & \big[f(\ulcorner\Theta\urcorner)\downarrow \wedge \boldsymbol{\omega}\text{-}\mathsf{Con}_Q\big(f(\ulcorner\Theta\urcorner)\big) \wedge \big(x = f(\ulcorner\Theta\urcorner) \vee x = \mathsf{q}\big)\big] \quad &\bigvee\\
& \big[f(\ulcorner\Theta\urcorner)\downarrow \wedge \neg\boldsymbol{\omega}\text{-}\mathsf{Con}_Q\big(f(\ulcorner\Theta\urcorner)\big) \wedge \big(x = \neg f(\ulcorner\Theta\urcorner) \vee x = \mathsf{q}\big)\big] \quad &\bigvee\\
& (x = \mathsf{q}).
\end{aligned}
$$

Now, if $f(\ulcorner\Theta\urcorner)\uparrow$ then $\Theta(x) \equiv (x = \mathsf{q})$ and so $\mathcal{T}_\Theta = Q$ is an n-consistent extension of Q, whence $f(\ulcorner\Theta\urcorner)\downarrow$; contradiction. Thus, $f(\ulcorner\Theta\urcorner)\downarrow$. If $Q \cup \{f(\ulcorner\Theta\urcorner)\}$ is ω-consistent then we have $\Theta(x) \equiv (x = f(\ulcorner\Theta\urcorner) \vee x = \mathsf{q})$ and so $\mathcal{T}_\Theta = Q \cup \{f(\ulcorner\Theta\urcorner)\}$ is an n-consistent extension of Q, whence $f(\ulcorner\Theta\urcorner)$ should be independent from it; contradiction. So, $Q \cup \{f(\ulcorner\Theta\urcorner)\}$ is not ω-consistent; then by [3, Theorem 21] (which states that for any ω-consistent theory S and any sentence X either $S \cup \{X\}$ or $S \cup \{\neg X\}$ is ω-consistent) the theory $Q \cup \{\neg f(\ulcorner\Theta\urcorner)\}$ should be ω-consistent. But in this case we have $\Theta(x) \equiv (x = \neg f(\ulcorner\Theta\urcorner) \vee x = \mathsf{q})$ and so $\mathcal{T}_\Theta = Q \cup \{\neg f(\ulcorner\Theta\urcorner)\}$ is an n-consistent extension of Q, whence $f(\ulcorner\Theta\urcorner)$ should be independent from it; contradiction again. Thus there can be no such $\emptyset^{(n)}$-recursive function.   ❑

Remark 3.6 (Optimality of Theorem 3.5) Even though, by Theorem 3.5, there does not exist any $\emptyset^{(n)}$-recursive function (for $n > 2$) which can output an independent Π_{n+1}-sentence for a given Σ_{n+1}-definable and n-consistent extension of Q, there indeed exists some $\emptyset^{(n+1)}$-recursive function which can find such an independent Π_{n+1}-sentence (for a given Σ_{n+1}-definition of an n-consistent extension of Q): By having an access to the oracle $\emptyset^{(n+1)}$ for a given $\mathsf{Ax}_T \in \Sigma_{n+1}$, provability (or unprovability) in T of a given sentence is decidable; thus (since by Corollary 2.11 there must exist some Π_{n+1}-sentence independent from the theory T) by an exhaustive search through all the Π_{n+1}-sentences such an independent sentence can be eventually found.   ✧

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4 Conclusions

Summing up, Gödel first incompleteness theorem in its semantic form, which states the Π_1-incompleteness of any sound and Σ_1-definable extension of Q, can be generalized to show that any sound and Σ_n-definable extension of Q is Π_n-incomplete. Also, Gödel's original first incompleteness theorem, which is equivalent to the Π_1-undecidability of any Σ_1-sound and Σ_1-definable extension of Q, can be generalized to show that no Σ_n-sound and Σ_n-definable extension of Q is Π_n-deciding (here actually Σ_{n-1}-soundness suffices by Rosser's Trick). Finally, Rosser's incompleteness theorem, which states the Π_1-undecidability of any consistent and Σ_1-definable extension of Q, cannot be generalized to definable theories, not even to Π_1-definable ones. Concluding, we have the following table for $n>1$ which shows our results in a viewable perspective:

Gödel's 1st (Semantic)	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_1$ & T is (Σ_∞)Sound $\implies$	$T \notin \Pi_1-$Complete
Theorem 2.1	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & T is (Σ_∞)Sound $\implies$	$T \notin \Pi_n-$Complete
Gödel's 1st	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_1$ & T is Σ_1-Sound $\implies$	$T \notin \Pi_1-$Deciding
Corollary 2.8	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & T is Σ_n-Sound $\implies$	$T \notin \Pi_n-$Deciding
Gödel–Rosser	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_1$ & T is Σ_0-Sound $\implies$	$T \notin \Pi_1-$Deciding
Corollary 2.9	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & T is Σ_{n-1}Sound $\implies$	$T \notin \Pi_n-$Deciding
Corollary 3.4(1)	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & T is Σ_{n-2}Sound $\not\implies$	$T \notin$ Complete
Gödel's 1st	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_1$ & 1-$\mathsf{Con}(T)$ $\implies$	$T \notin \Pi_1-$Deciding
Corollary 2.10	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & n-$\mathsf{Con}(T)$ $\implies$	$T \notin \Pi_n-$Deciding
Gödel–Rosser	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_1$ & $\mathsf{Con}(T)$ $\implies$	$T \notin \Pi_1-$Deciding
Corollary 2.11	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & $(n-1)$-$\mathsf{Con}(T)$ $\implies$	$T \notin \Pi_n-$Deciding
Corollary 3.4(1)	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Sigma_n$ & $(n-2)$-$\mathsf{Con}(T)$ $\not\implies$	$T \notin$ Complete

To complete the picture here are the Π version of the results for $m>0$:

Theorem 2.4	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Pi_m$ & T is Σ_m-Sound $\implies$	$T \notin \Pi_{m+1}-$Deciding
Corollary 3.4(2)	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Pi_m$ & T is Σ_{m-1}Sound $\not\implies$	$T \notin$ Complete
Corollary 2.6	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Pi_m$ & m-$\mathsf{Con}(T)$ $\implies$	$T \notin \Pi_{m+1}-$Deciding
Corollary 3.4(2)	$Q \subseteq T$ & $\mathsf{Axioms}_T \in \Pi_m$ & $(m-1)$-$\mathsf{Con}(T)$ $\not\implies$	$T \notin$ Complete

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